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ON A VARIATIONAL INEQUALITY FOR A SHALLOW SHELL OPERATOR WITH A CONSTRAINT ON THE BOUNDARY*

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A variational inequality that describes shell contact with a rigid stamp on the boundary is investigated. A non-negative measure characrerizing the action of the stamp on the shell is constructed on subsets of the boundary. The regularity of the solution is established.

A number of results referring to the investigation of variational inequalities describing contact problems for elastic bodies under unilateral contiguity conditions has been obtained at this time. In particular, the contact problem for a three-dimensional elastic body (Signorini's problem) was considered in /1/. Its more general formulation for the case of a stamp whose surface does not agree with the elastic body boundary is proposed /2/ and then investigated /3/. Unilateral contact problems were considered for plates /4/ and shells /5/ with a constraint within the domain.

Let us consider a boundary value problem for linear shallow shell equations with conditions on part of the boundary Γ_0 having the form

$$u_3 \geqslant 0,\ T(u_3) \geqslant 0,\ u_3 T(u_3) = 0,\ M(u_3) = 0 \tag{1}$$

$$-U_n \geqslant 0,\ -N_n \geqslant 0,\ U_n N_n = 0,\ N_\tau = 0 \tag{2}$$

Here $M(u_3)$, $T(u_3)$ are the bending moment and transverse force, $U = (u_1, u_2)$, $U_n = u_i n_i$, u_1, u_2, u_3 are, respectively, the tangential and normal displacements of points of the shell, $n = (n_1, n_2)$ is the external normal to the boundary, $N_n = N_{ij} n_j n_i$, N_{ij} is the force in the middle surface, N_τ is the tangential component of the force vector on the boundary. Summation is over repeated subscripts i, j. The boundary conditions formulated correspond to unilateral shell contact on the boundary with a rigid stamp and allow separation of the shell points from the stamp in both the $x_1 x_2$ plane and in a direction normal to the middle surface. The separation condition is ensured by the possibility of satisfying the strict inequalities $u_3 > 0$ or $-U_n > 0$. In this case $T(u_3) = 0$, or respectively, $N_n = 0$. If $T(u_3) > 0$ or $-N_n > 0$, then, correspondingly, $u_3 = 0$ and $U_n = 0$.

We will introduce a number of notations and construct an exact formulation of the problem. Let $\Omega \subset R^2$ be a bounded domain with the smooth boundary Γ represented in the form of the union of two parts: $\Gamma = \Gamma_0 \cup \Gamma_1$. For simplicity, we assume that Γ_0 and Γ_1 are arcs where the length of Γ_1 is greater than zero. We let $H_{\Gamma_1}^1(\Omega)$ denote the Sobolev space obtained by the closure of smooth functions equal to zero in the neighbourhood of Γ_1 in $H^1(\Omega)$. The space $H_{\Gamma_1}^2(\Omega)$ is defined similarly. Also let $H(\Omega) = H_{\Gamma_1}^1(\Omega) \times H_{\Gamma_1}^1 \times H_{\Gamma_1}^2(\Omega)$, $\|\cdot\|_s$ be the norm in $H^s(\Omega)$.

We consider the shell energy functional

$$\Pi(\omega) = \Pi_1(\omega) - 2\int_\Omega F\omega\, dx, \quad \omega = (u_1, u_2, u_3) \tag{3}$$

$$\Pi_1(\omega) = B(u_3, u_3) + \int_\Omega \left\{\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\sigma\varepsilon_{11}\varepsilon_{22} + \frac{1}{2}(1-\sigma)\varepsilon_{12}^2\right\} dx$$

$$\varepsilon_{11} = u_{1x_1} + k_{11}u_3,\ \varepsilon_{22} = u_{2x_2} + k_{22}u_3,\ \varepsilon_{12} = u_{1x_2} + u_{2x_1}$$

Here $F = (f_1, f_2, f_3) \in L^2(\Omega)$ is the vector of the given forces, $k_{11}, k_{22} \in C^1(\overline{\Omega})$ are the curvatures, σ is Poisson's ratio, and $x = (x_1, x_2) \in \Omega$. The bilinear form $B(\cdot, \cdot)$ is defined below by /7/.

Furthermore, we introduce the closed convex set in $H(\Omega)$

$$K = \{\omega = (U, u_3) \in H(\Omega) \mid u_3 \geqslant 0,\ -U \in 0 \quad \text{on} \quad \Gamma_0\}$$

and we consider the problem of minimizing the energy functional $\Pi(\omega)$ in the set K. It is equivalent to solving the variational inequality

$$\omega \in K: \langle \Pi'(\omega), \chi - \omega\rangle \geqslant 0,\ \forall\chi \geqslant K \tag{4}$$

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Here $\Pi'(\omega)$ is the gradient of the functional Π at the point ω.

At can be shown that a solution of the problem exists.

We obtain from inequality (4) that the following equations will be satisfied in the distribution sense in the domain Ω:

$$\Delta^2 u_3 + k_{11}N_{11} + k_{22}N_{22} = f_3; \quad -\partial N_{ij}/\partial x_j = f, \quad i = 1, 2$$
$$N_{11} = \varepsilon_{11} + \sigma\varepsilon_{22}, \; N_{22} = \varepsilon_{22} + \sigma\varepsilon_{11}, \; N_{12} = {}^1/_2\,(1-\sigma)\,\varepsilon_{12} \tag{5}$$

To prove this fact it is sufficient to substitute $\omega + \omega_0$ as χ into inequality (4), where $\omega_0 \in C_0^\infty(\Omega)$ is an arbitrary function.

Furthermore, we write the formulas for the moment and transverse force

$$M(u_3) = \sigma\Delta u_3 + (1-\sigma)\frac{\partial^2 u_3}{\partial n^2}$$
$$T(u_3) = -\frac{\partial}{\partial n}\Delta u_3 - (1-\sigma)\frac{\partial}{\partial\tau}\frac{\partial^2 u_3}{\partial n\partial\tau} \tag{6}$$

Here $\tau = (-n_2, n_1)$ is a vector tangent to Γ. We also introduce a bilinear form that takes part in the representation of the energy functional $\Pi(\omega)$

$$B(\varphi,\psi) = \int_\Omega \{\varphi_{x_1x_1}\psi_{x_1x_1} + \varphi_{x_2x_2}\psi_{x_2x_2} + \sigma(\varphi_{x_1x_1}\psi_{x_2x_2} + \varphi_{x_2x_2}\psi_{x_1x_1}) + 2(1-\sigma)\varphi_{x_1x_2}\psi_{x_1x_2}\}\,dx \tag{7}$$

The formal foundation (assuming sufficient regularity of the solution) of the fact that the boundary conditions (1) and (2) will be satisfied on Γ_0 can be obtained by using Green's formula for a biharmonic operator and the operator of the plane problem of elasticity theory.

An exact mathematical meaning can be given to the boundary conditions (1) and (2). To do this it is necessary to use theorems about traces. It follows from the first equation in (5) that $\Delta^2 u_3 \in L^2(\Omega)$. Moreover, from the fact that u_3 belongs to the space $H^2(\Omega)$, we have $u_3 \in H^{3/2}(\Gamma)$, $\partial u_3/\partial n \in H^{1/2}(\Gamma)$ on the boundary Γ. According to /6/, for the elements of the space $\{w \in H^2(\Omega) \mid \Delta^2 w \in L^2(\Omega)\}$ it is possible to determine $M(w) \in H^{-1/2}(\Gamma)$, $T(w) \in H^{-3/2}(\Gamma)$, where the generalized Green's formula

$$B(w,\psi) = \langle\Delta^2 w,\psi\rangle + \langle T(w),\psi\rangle_{3/2} + \left\langle M(w), \frac{\partial\psi}{\partial n}\right\rangle_{1/2}, \quad \forall\psi\in H^2(\Omega) \tag{8}$$

holds.

Here $H^{-s}(\Gamma)$ is the space that is topologically conjugate to the space $H^s(\Gamma)$, and the brackets $\langle\cdot,\cdot\rangle_s$ denote the duality between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$. The conditions on the boundary operators M, T necessary for the correctness of this result are confirmed in /7/. Thus the quantities $u_3T(u_3)$ in (1) allows of accurate interpretation.

It follows from (5) that $\partial N_{ij}/\partial x_j \in L^2(\Omega)$. As is shown in /8/, for the function $\varphi = (\varphi_1, \varphi_2)$ satisfying the inclusions φ, $\operatorname{div}\varphi \in L^2(\Omega)$, $\varphi_i n_i \in H^{-1/2}(\Gamma)$ can be defined on the boundary. Consequently, $N_{ij}n_j \in H^{-1/2}(\Gamma)$. We hence obtain $N_n \in H^{-1/2}(\Gamma)$. Taking account of the inclusion $U_n \in H^{1/2}(\Gamma)$, the product U_nN_n in (2) can also be given an exact meaning.

Non-negative measures μ_1, μ_2 characterising the stamp reaction on the shell are constructed below in subsets of the boundaries $\Gamma_0 \setminus \partial\Gamma_0$. The measure μ_2 characterises the reaction of the stamp in the x_1x_2 plane in the normal direction to the boundary, and μ_1 in an orthogonal direction to the shell middle surface.

We introduce the space $C_0(\Gamma_0)$ of finite functions continuous in Γ_0 with the following convergence. We assume that $\varphi_n \to \varphi$ if φ_n converges uniformly to φ and the carriers of all φ_n belong to a fixed compactum $B \subset \Gamma_0 \setminus \partial\Gamma_0$.

Theorem 1. The non-negative measures μ_1, μ_2 for which the representation

$$\langle\Pi'(\omega),\chi\rangle = \int_{\Gamma_0} v_3\,d\mu_1 - \int_{\Gamma_0} V_n\,d\mu_2, \quad \forall\chi = (V, v_3) \in H(\Omega) \cap C_0(\Gamma_0) \tag{9}$$

holds can be determined in the σ-algebra of Borel subsets of the boundary $\Gamma_0 \setminus \partial\Gamma_0$.

Proof. We first note the following fact. Let $\chi_0 = (0, 0, v_3) \in H(\Omega)$ and $v_3 \geqslant 0$ on Γ_0. Then

$$\langle\Pi'(\omega)), \chi_0\rangle \geqslant 0 \tag{10}$$

To prove this assertion it is sufficient to substitute the function $\omega + \chi_0$ as χ into the inequality (4). Furthermore, let $v_3 \in H_{\Gamma_1}{}^2(\Omega) \cap C_0(\Gamma_0)$ and $v_3{}^*$ be the trace of this function on Γ_0. A linear manifold of all such functions on Γ_0 will be denoted by V. We define the linear functional on V by the formula

$$L(v_3{}^*) = \langle\Pi'(\omega),\chi\rangle, \quad \chi = (0, 0, v_3)$$

The functional L is defined uniquely by this formula. In fact, if $v_3{}^{1*} = v_3{}^{2*}$, then according to (10) we have $L(v_3^{1*}) = L(v_3^{2*})$. Furthermore, we select an arbitrary element $v_3^* \in C_0^2(\Gamma_0); C_0^2(\Gamma_0)$ of the space of finite functions on Γ_0 that have two continuous derivatives. The function v_3^* can be continued to zero on the whole boundary Γ and then continued within the domain Ω such that it becomes a function of the class $H_{\Gamma_1}^2(\Omega)$. This means that the lineal V contains all functions from $C_0^2(\Gamma_0)$. In continuity the functional L is continued on $C_0(\Gamma_0)$. At the same time an arbitrary linear positive functional on $C_0(\Gamma_0)$ is determined by the measure

$$L(v_3^*) = \int_{\Gamma_0} v_3^* \, d\mu_1$$

For a function $\chi \in H(\Omega) \cap C_0(\Gamma_0)$ of the form $(0, 0, v_3)$ this denotes the validity of the representation

$$\langle \Pi'(\omega), \chi \rangle = \int_{\Gamma_0} v_3 \, d\mu_1 \tag{11}$$

Furthermore, we note that the second and third equations in (5) with the boundary conditions (2) are the analogue of the two-dimensional Signorini problem. The fact that the forces N_{ij} depend on the deflection u_3 and the curvatures k_{11}, k_{22} is not essential. Consequently, the measure μ_2 can be constructed in the same way as in /3/. Therefore, for any function $\chi \in H(\Omega) \cap C_0(\Gamma_0)$ of the form $\chi = (v_1, v_2, 0)$ the following equality holds:

$$\langle \Pi'(\omega), \chi \rangle = -\int_{\Gamma_0} V_n \, d\mu_2, \quad V = (v_1, v_2) \tag{12}$$

By virtue of the additivity of (11) and (12) we obtain the representation (9).

The measures constructed take finite values in all the compacta $B \subset \Gamma_0 \setminus \partial\Gamma_0$. The properties of the measure μ_2 depend mainly on the regularity of the function U. In particular, available results on the smoothness of the solution of the Signorini problem enable us to prove absolute continuity of the measure μ_2 relative to the Lebesgue measure on $\Gamma_0 \setminus \partial\Gamma_0$. Namely, for an arbitrary point $x \in \Gamma_0 \setminus \partial\Gamma_0$ there exists a neighbourhood Ω_0 such that $U \in H^2(\Omega_0 \cap \Omega)$. The density of the measure μ_2 turns out to be equal to $-N_n$, where $N_n \in H_{\mathrm{loc}}^{1/2}(\Gamma_0 \setminus \partial\Gamma_0)$. As regards the measure μ_1, its properties are then determined by the smoothness of the function u_3.

Theorem 2. For an arbitrary point $x^0 \in \Gamma_0 \setminus \partial\Gamma_0$ a neighbourhood Ω_0 exist to such that $u_3 \in H^3(\Omega_0, \cap \Omega)$.

Proof. We place the origin at the point x^0 by considering the direction of the x_2 axis to be the same as the direction of the external normal to the boundary Γ. For simplicity, we still assume that a section of the boundary Γ_0 near x^0 is rectilinear. We set

$$d_\tau h(x) = \tau^{-1}[h(x + \tau e_1) - h(x)], \quad \Delta_\tau = -d_{-\tau} d_\tau$$

Here $\tau > 0$, and e_1 is the unit direction of the x_1 axis. Let R_δ denote a circle of radius δ with centre at the point x^0. Let $\varphi \in C_0^\infty(R_\delta), \varphi \equiv 1$ on $R_{\delta/2}, 0 \leqslant \varphi \leqslant 1$, and $r < \delta/2$. Then the function $u_{3\tau} = u_3 + {}^1/_2\tau^2\varphi^2\Delta_\tau u_3$ satisfies the inequality $u_{3\tau} \geqslant 0$ on Γ_0. Indeed, by considering the parameter δ to be sufficiently small, we have for $x \in \Gamma_0$

$$u_{3\tau}(x) = (1 - \varphi^2(x))u_3(x) + {}^1/_2\varphi^2(x)(u_3(x + \tau e_1) + u_3(x - \tau e_1)) \geqslant 0$$

This means that $(u_1, u_2, u_{3\tau}) \in K$. We substitute $(u_1, u_2, u_{3\tau})$ as a test into inequality (4). We obtain

$$B(u_3, \varphi^2\Delta_\tau u_3) - \langle f_3 - k_{11}N_{11} - k_{22}N_{22}, \varphi^2\Delta_\tau u_3 \rangle \geqslant 0 \tag{13}$$

The following chain is valid for which the difference between two successive terms is either zero or has a quantity as upper bound that is contained on the right-hand side of the inequality (14) obtained below:

$$B(u_3, \varphi^2\Delta_\tau u_3) \to B(\varphi u_3, \Delta_\tau \varphi u_3) \to B(\varphi u_3, -d_{-\tau} d_\tau \varphi u_3) \to$$
$$-B(d_\tau \varphi u_3, d_\tau \varphi u_3)$$

The second component in (13) is estimated more simply. It therefore follows from (13)

$$\| d_\tau(\varphi u_3) \|_2^2 \leqslant c \, \{\| f_3 \|_0^2 + \| u_1 \|_1^2 + \| u_2 \|_1^2 + \| u_3 \|_2^2 + \| d_\tau(\varphi u_3) \|_2 \| u_3 \|_2\} \tag{14}$$

Here the constant c is independent of τ. We hence obtain the boundedness of $\| d_\tau(\varphi u_3) \|_2$ uniformly in τ. This means that all three derivatives of u_3, with the exception of $\partial^3 u_3/\partial x_2^3$, belong to $L^2(R_{\delta/2} \cap \Omega)$. We write the first equation in (5) in the form

$$\partial^4 u_3/\partial x_2^4 = g \tag{15}$$

It follows from what was proved that $g \in H^{-1}(R_{\delta/2} \cap \Omega)$. At the same time, we have $\partial^3 u_3/\partial x_2^3 \in H^{-1}(R_{\delta/2} \cap \Omega)$ from the fact that u_3 belongs to the space $H^2(\Omega)$. Together with (15) this yields $\partial^3 u_3/\partial x_2^3 \in L^2(R_{\delta/2} \cap \Omega)$, which indeed proves the theorem in this case. The following fact is used here. If $\varphi, \varphi_{x_i} \in H^{-1}(\Omega)$, then $\varphi \in L^2(\Omega)$ (see /9/). If the section of Γ_0 near the point x^0 is not rectilinear, then it is possible to make a change of variable with the unit Jacobian

$y_1 = x_1, y_2 = x_2 - \alpha(x_1)$. Here $x_2 = \alpha(x_1)$ is the equation of the boundary near the point x°. The nature of the reasoning performed in this case is analoguous to that presented above.

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MULTIPLE EIGENVALUES IN OPTIMIZATION PROBLEMS*

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The problem of maximizing the minimum eigenvalue of a selfadjoint matrix operator is considered. The case when the optimum eigenvalue is multiple, i.e. the problem of optimization is discontinuous, is investigated. This problem has interesting applications in the optimum design of constructions /1-6/. The necessary conditions for a local maximum of the eigenvalue of arbitrary multiplicity p with an **isoperimetric** limit are obtained. The paper generalizes the results obtained in /7, 8/ for the single and double case.

Consider the eigenvalue problem

$$A[h]u = \lambda B[h]u \tag{1}$$

Here $A[h]$ and $B[h]$ are positive-definite symmetric $m \times m$ matrices with coefficients $a_{ij}(h)$ and $b_{ij}(h)$, which depends continuously on the components of the vector of the parameters h of dimensions n, u is an eigenvector of dimensions m, and λ is an eigenvalue.

Problem (1) has a complete system of eigenvectors u^i $(i = 1, 2, \ldots, m)$ and a sequence of eigenvalues λ_i $(i = 1, 2, \ldots, m)$ corresponding to this system; we will assume that the orthogonality condition is satisfied

$$(B[h]u^i, u^j) = \delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta. Here and henceforth the parenthesis denote the scalar product of vectors.

We will formulate the optimization problem as follows: it is required to obtain the vector of the parameters $h = (h_1, h_2, \ldots, h_n)$ for which the minimum eigenvalue λ_1 of problem (1) reaches a maximum value under the conditions

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